RANDOM-SET THEORY AND ITS APPLICATIONS TO WIRELESS COMMUNICATIONS

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MOTIVATION

multiuser detection

A multiuser system is described by the set

$$\mathbf{X}_t = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}, \quad k = 0, 1, \ldots$$

where k is the number of active interferers, and

$\mathbf{x}_i$ are the state vectors of the individual interferers

($k=0$ yields the empty set, and corresponds to no interferer)

multiuser detection

If the number of interferers is not known a priori (as it is usually the case) then $\mathbf{X}_t$ is a *random set*, that is, a set *whose randomness is not only in the elements, but also in the number of elements*.

DEFINE RANDOM SETS

random set theory

- RST is a probability theory of finite sets that exhibit randomness not only in each element, but also in the number of elements

random set theory

Consider a topological space $\mathbb{S}$

(technically, a *locally compact Hausdorff separable* space
--- important special case: $\mathbb{R}^d$)

random set theory

We define a *random closed set* as a map between a sample space and the family of closed subsets of $\mathbb{S}$

$$\mathbf{X} : \Omega \to \mathfrak{F}(\mathbb{S})$$

random set theory

More formally,

$$\mathbf{X} : \Omega \rightarrow \mathfrak{F}(\mathbb{S})$$

is a random closed set if, for every compact $K \in \mathbb{S}$, we have

$$\{\omega : \mathbf{X} \cap K \neq \emptyset\} \in \mathfrak{F}(\mathbb{S})$$

(that is, observing $\mathbf{X}$ one can always say if $\mathbf{X}$ hits or misses any given compact set K)

random set theory

EXAMPLE 1

$\mathbf{X} = (-\infty, \xi]$ (ξ a real random variable) is a random closed set on the real line. In fact,

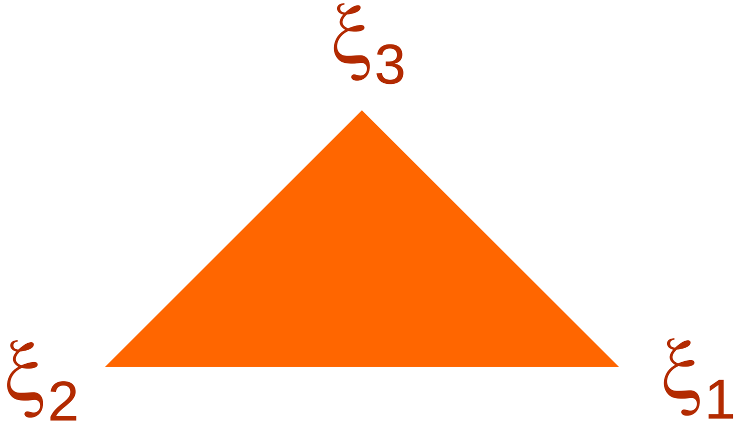
$$\{\mathbf{X} \cap K \neq \emptyset\} = \{\xi \geq \inf K\}$$

a measurable event for every compact K

random set theory

EXAMPLE 2

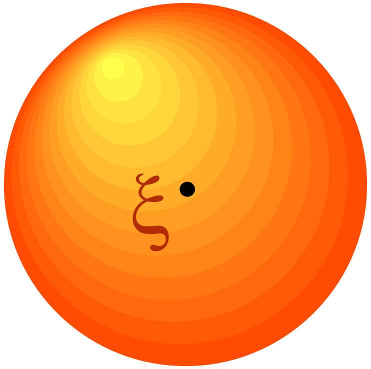
The triangle with vertices ξ_1, ξ_2, ξ_3
(ξ_i random 2-vectors)
is a random closed set



random set theory

EXAMPLE 3

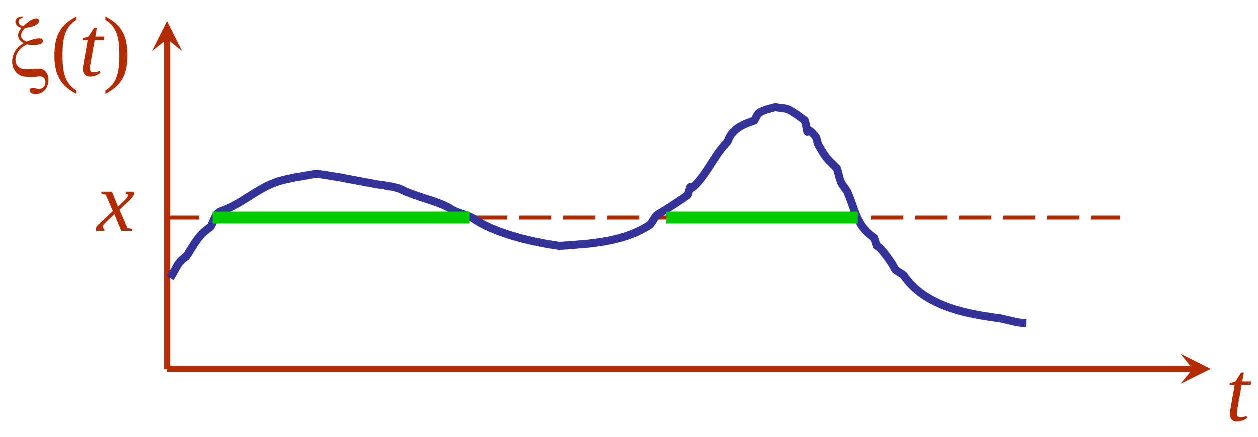
The sphere with radius η centered at ξ (η a positive random variable, ξ a random 3-vector) is a random closed set



random set theory

EXAMPLE 4

The level set $\mathbf{X} = \{t : \xi(t) \geq x\}$,
$\xi(t)$ a random process, is a random set



A PROBABILITY THEORY OF RANDOM SETS

random set theory

Define the *Belief Function*

$$\beta_{\mathbf{X}}(\mathbf{C}) \triangleq \mathbb{P}(\mathbf{X} \subseteq \mathbf{C})$$

where $\mathbf{C}$ is a closed subset of $\mathbb{S}$

random set theory

Decomposition of a belief function into a sum of simpler belief functions:

$$\beta_{\mathbf{X}}(S) \triangleq \mathbb{P}(\mathbf{X} \subseteq S)$$

$$= \sum_{k=0}^{\infty} \mathbb{P}(|\mathbf{X}| = k)\mathbb{P}(\mathbf{X} \subseteq S \mid |\mathbf{X}| = k)$$

random set theory

A special case shows that the belief function generalizes standard probability measures. For $\mathbf{X} = \{\mathbf{x}\}$, $\mathbf{x}$ a random vector:

$$\beta_{\mathbf{X}}(\mathbf{C}) \triangleq \mathbb{P}(\mathbf{X} \subseteq \mathbf{C}) = \mathbb{P}(\mathbf{x} \in \mathbf{C}) = \mathbb{P}_{\mathbf{x}}(\mathbf{C})$$

random set theory

However, belief functions are *not probability measures*: $\mathbf{C}_1 \cap \mathbf{C}_2 = \emptyset$ implies

$$\mathbb{P}(\mathbf{x} \in \mathbf{C}_1 \cup \mathbf{C}_2) = \mathbb{P}_{\mathbf{x}}(\mathbf{C}_1) + \mathbb{P}_{\mathbf{x}}(\mathbf{C}_2)$$

but

$$\beta_{\mathbf{X}}(\mathbf{C}_1 \cup \mathbf{C}_2) > \beta_{\mathbf{X}}(\mathbf{C}_1) + \beta_{\mathbf{X}}(\mathbf{C}_2)$$

example of belief function

("Singleton-or-empty" sets).

Let

$$\mathbf{X} = \begin{cases} \{\mathbf{x}\}, \ \mathbf{x} \in \mathbb{R}^d & \text{with prob. } 1-q \\ \emptyset, & \text{with prob. } q \end{cases}$$

We obtain

$$\begin{aligned} \beta_{\mathbf{X}}(\mathbf{C}) &\triangleq \mathbb{P}\{\mathbf{X} \subseteq \mathbf{C}\} \\ &= \mathbb{P}\{\mathbf{X} = \emptyset\} + \mathbb{P}\{\mathbf{x} \in \mathbf{C}, \mathbf{X} \neq \emptyset\} \\ &= q + (1-q)\mathbb{P}_{\mathbf{x}}(\mathbf{C}) \end{aligned}$$

set integral

Define the *set integral* of Φ at S:

$$\int_S \Phi(Z)\,\delta Z \triangleq \sum_{k=0}^{\infty} \int_{S^k} \Phi_k(\xi_1, \ldots, \xi_k)\, d\lambda(\xi_1) \cdots \delta\lambda(\xi_k)$$

where S is a closed set, λ a measure,

$$\Phi_k(\xi_1, \ldots, \xi_k) \triangleq \begin{cases} \Phi(\{\xi_1, \ldots, \xi_k\}), & \xi_1, \ldots \xi_k \text{ distinct} \\ 0, & \text{otherwise} \end{cases}$$

and $\int_{S^0} \Phi(Z)\,\delta Z \triangleq \Phi(\emptyset)$

set derivative

Define the *set derivative* of Φ at $\{\xi\}$:

$$\frac{\delta\Phi}{\delta\xi}(T) \triangleq \lim_{B\to\{\xi\}} \frac{\Phi(T\cup B)-\Phi(T)}{\lambda(B)}$$

and

$$\frac{\delta\Phi}{\delta Z}(T) \quad \triangleq \quad \frac{\delta^k}{\delta\xi_1\cdots\delta\xi_k}\Phi(T), \quad Z\neq\emptyset$$

$$\frac{\delta\Phi}{\delta\emptyset}(T) \quad \triangleq \quad \Phi(T)$$

fundamental theorem

Set derivative and set integral are the inverse of each other.

RANDOM-SET ESTIMATION THEORY

random-set estimation theory

- The *Belief density* is the set derivative of the belief function:

$$p_{\mathbf{x}}(Z) \triangleq \frac{\delta \beta_{\mathbf{x}}}{\delta Z}(\emptyset)$$

- It can be used as a density in ordinary detection/estimation theory.

Bayesian estimators

A Bayesian set estimator is generated by

- Choosing a cost function $C(\mathbf{X}, \hat{\mathbf{X}})$
- Finding the estimator that minimizes the cost function

Bayesian estimators

Choosing a cost function may not be an easy task: think for example of a situation in which we must estimate the *number of interferers and their power* in a multiple-access environment.

Bayesian estimators

Bayesian set estimator I

- First, estimate the cardinality of the set:

$$\widehat{n} \triangleq \arg\max_{n} \mathbb{P}(|\mathbf{X}| = n) = \arg\max_{n} \int_{|\mathbf{X}|=n} p_{\mathbf{X}}(Z)\, \delta Z$$

- Next, find

$$\widehat{\mathbf{X}} = \arg\max_{\mathbf{X}} p(\mathbf{X} \mid |\mathbf{X}| = \widehat{n})$$

Bayesian estimators

Bayesian set estimator II

- Find

$$\hat{\hat{\mathbf{X}}} = \max_{n, \mathbf{X} : |\mathbf{X}| = n} p_{\mathbf{X}}(\mathbf{X}) \frac{c^n}{n!}$$

where c is a small constant, expressing how close $\mathbf{X}$ and its estimate must be to have cost = 0 (different values of c correspond to different cost functions, and hence to different estimators).

Bayesian estimators

Bayesian set estimator III

- Estimate first the cardinality of the set and the identities of its elements, then their continuous parameters
  using a posteriori expectations.

SYSTEM MODELING

modeling observations

$$f(\mathbf{y}_t \mid \mathbf{X}_t)$$

Description of measurement process (the “channel”)

modeling the dynamics

$$f(\mathbf{X}_t \mid \mathbf{X}_{t-1})$$

Evolution of random set with time (Markovian assumption)

Bayesian filtering equations

$$f(\mathbf{X}_t \mid \mathbf{y}_{1:t-1}) = \int f(\mathbf{X}_t \mid \mathbf{X}_{t-1})\, f(\mathbf{X}_{t-1} \mid \mathbf{y}_{1:t-1})\, \delta\mathbf{X}_{t-1}$$

$$f(\mathbf{X}_t \mid \mathbf{y}_{1:t}) \propto f(\mathbf{y}_t \mid \mathbf{X}_t)\, f(\mathbf{X}_t \mid \mathbf{y}_{1:t-1})$$

- Integrals are "set integrals" (the inverses of set derivatives)
- Closed form in the finite-set case
- Otherwise, use "particle filtering"

APPLICATIONS TO WIRELESS COMMUNICATIONS: MULTIUSER DETECTION

multiuser detection

A multiuser system with unknown number of interferers is described by the random set

$$\mathbf{X}_t = \{\mathbf{x}_1^{(t)}, \ldots, \mathbf{x}_{k(t)}^{(t)}\}, \quad k = 0, 1, \ldots$$

where k is the number of active interferers, and

$\mathbf{x}_i$ are the state vectors of the individual interferers

(k=0 corresponds to no interferer)

previous work

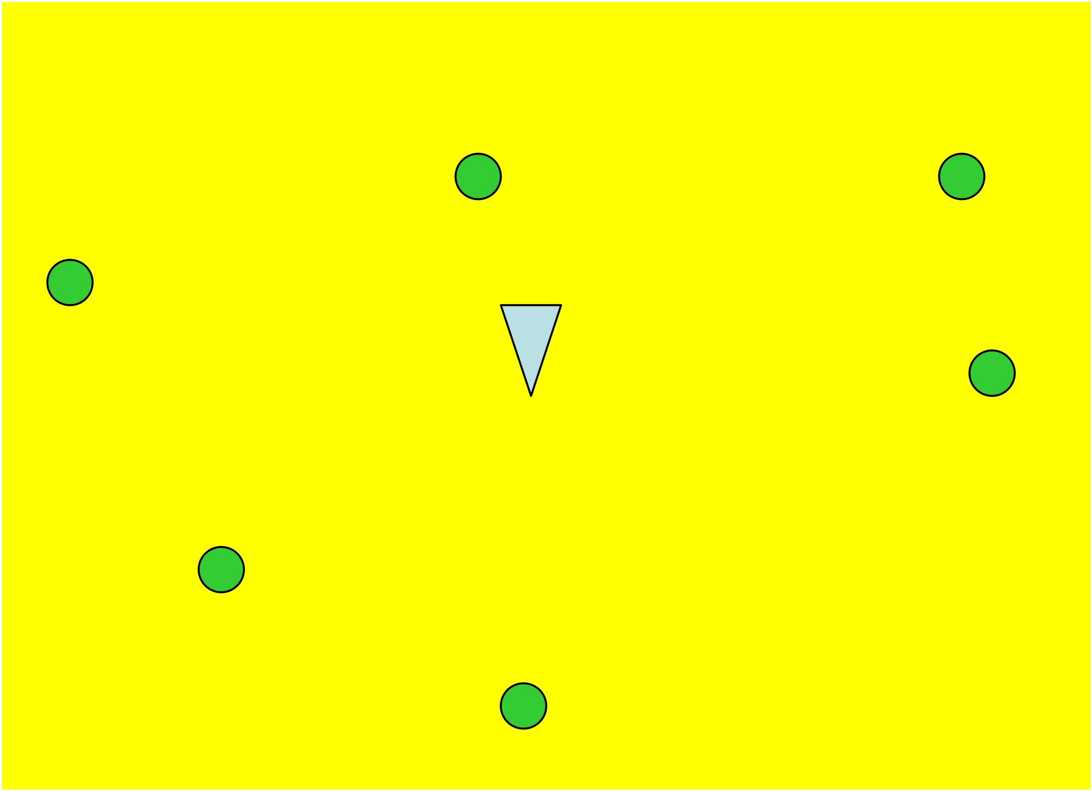
- Previous work (Mitra, Poor, Halford, Brandt-Pierce,...) focused on activity detection, addition of a single user.

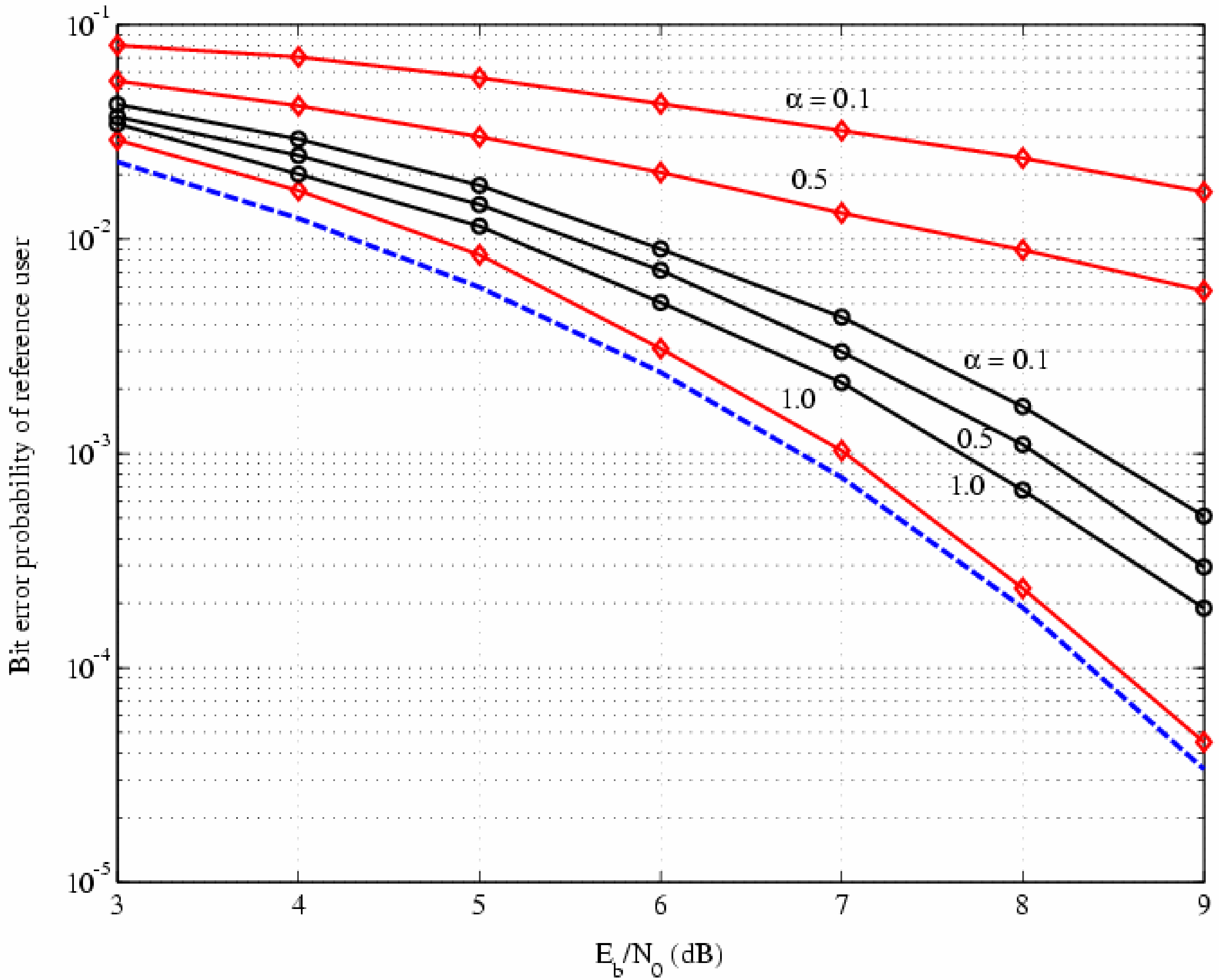
- It was recognized that certain detectors suffer from catastrophic error if a new user enter the system.

- Wu, Chen (1998) advocate a two-step detection algorithm:

  - *MUSIC algorithm estimates active users*
  - *MUD is used on estimated number of users*

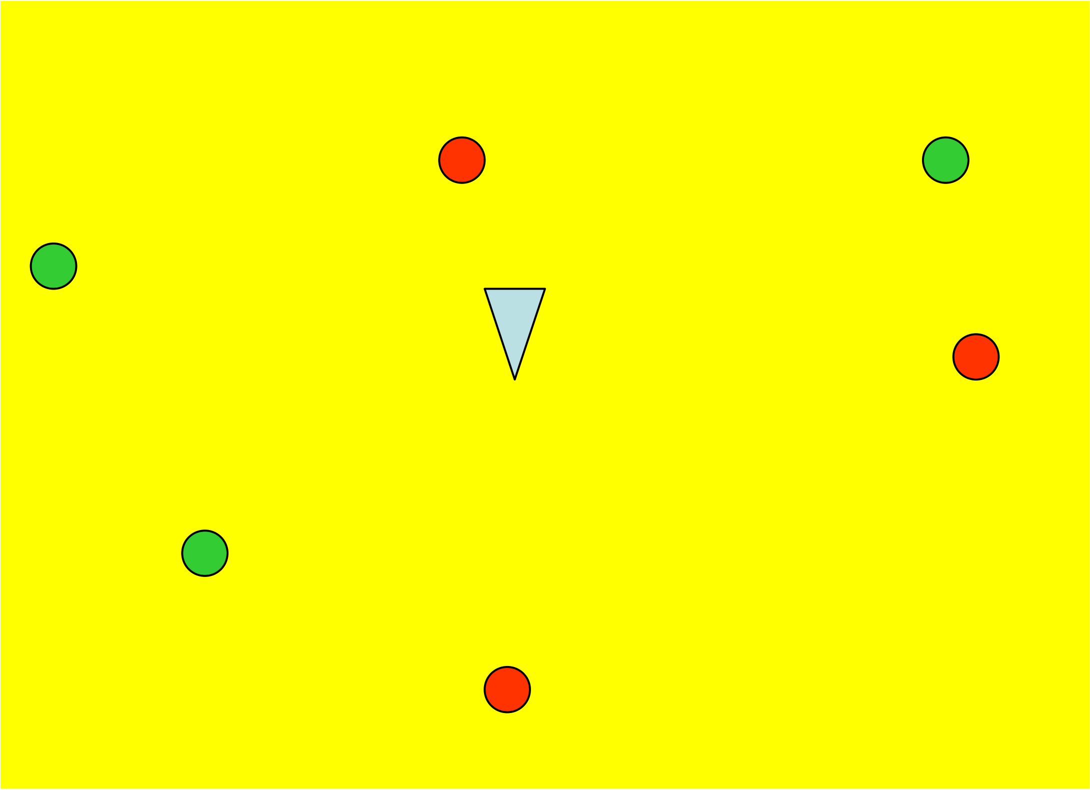
environment: static, deterministic

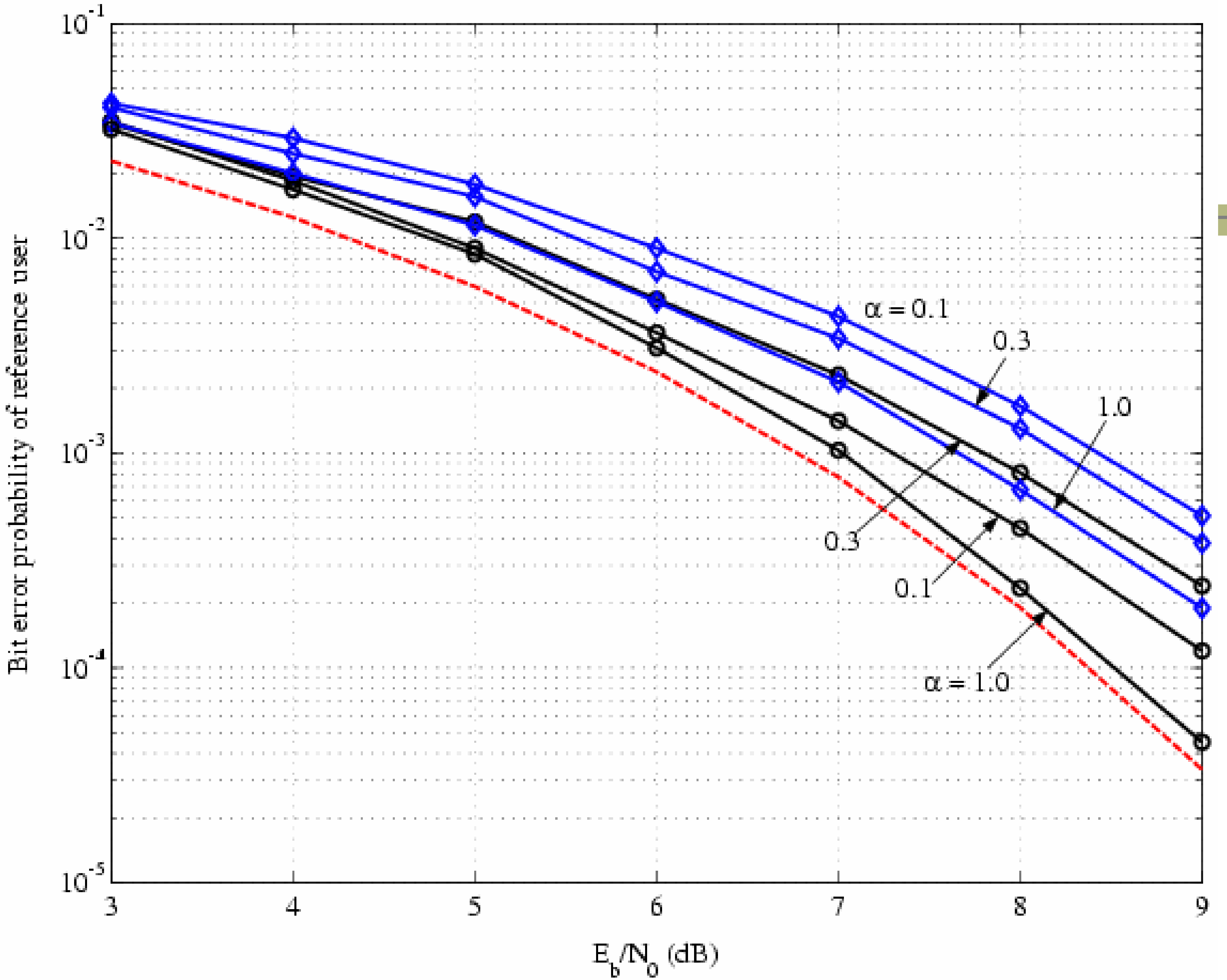




Static, random channel, 3 users:
Classic ML vs. joint ML detection of data and # of interferers

environment: static, random





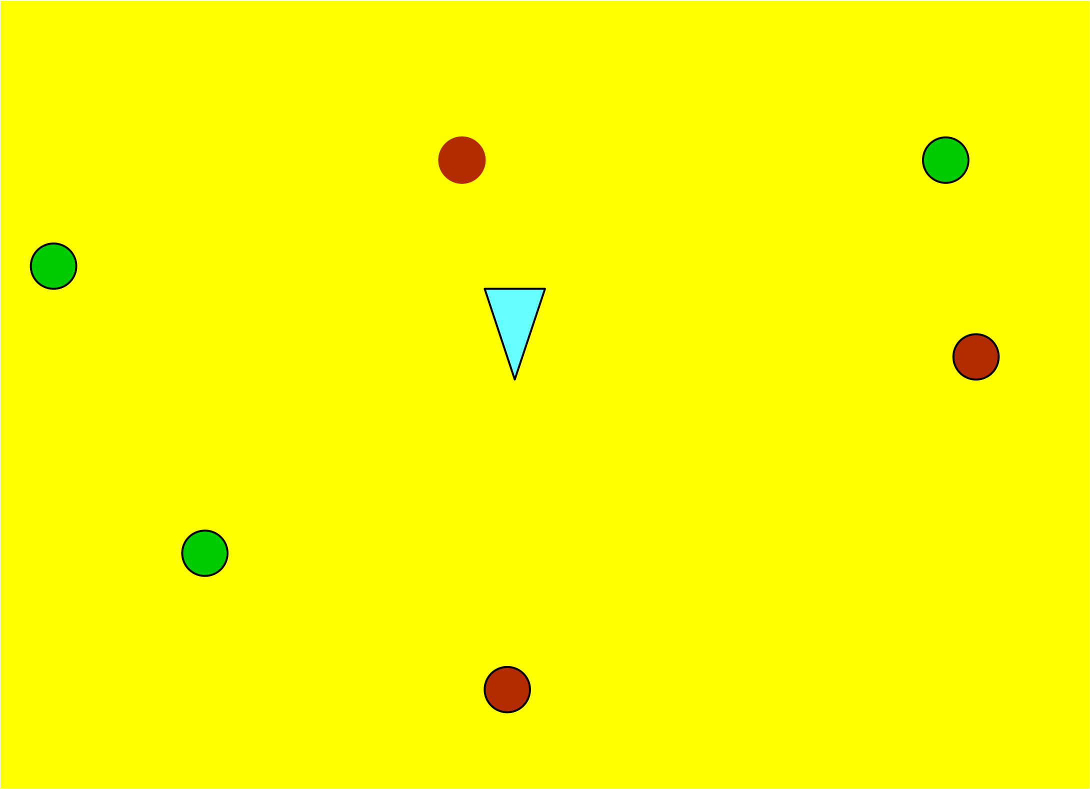
Static, random channel, 3 users:

Joint ML detection of data and # of interferes vs. MAP

mobility & wireless

(“La vie electrique,” ALBERT ROBIDA, French illustrator, 1892)

environment: dynamic, random



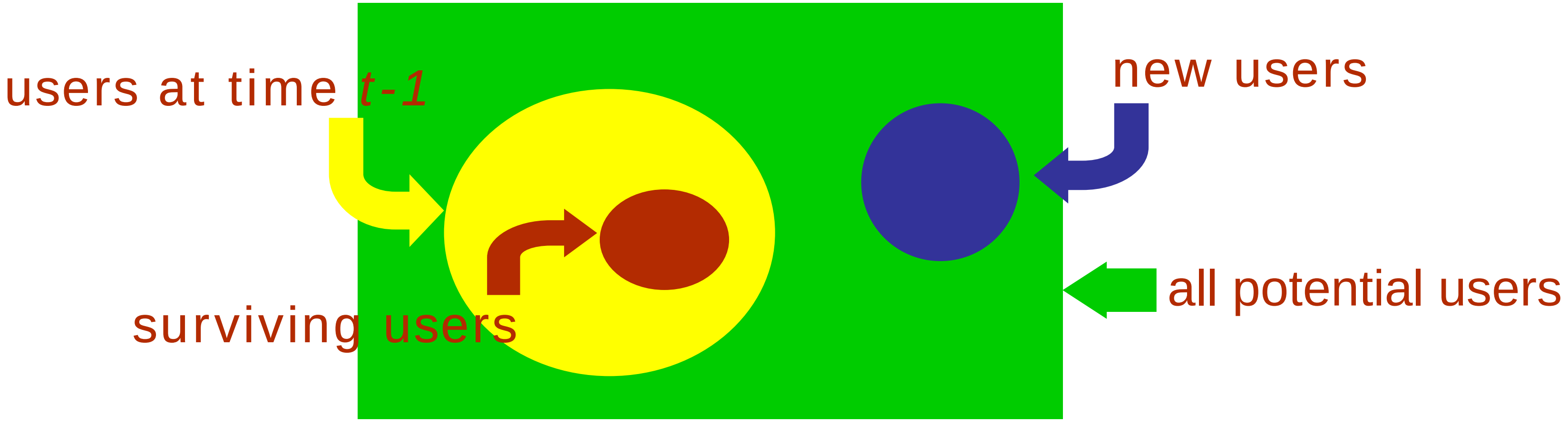
multiuser dynamics

$$\mathbf{X}_t = \mathbf{S}_t \cup \mathbf{N}_t$$

random set: users at time *t*

users surviving from time *t-1*

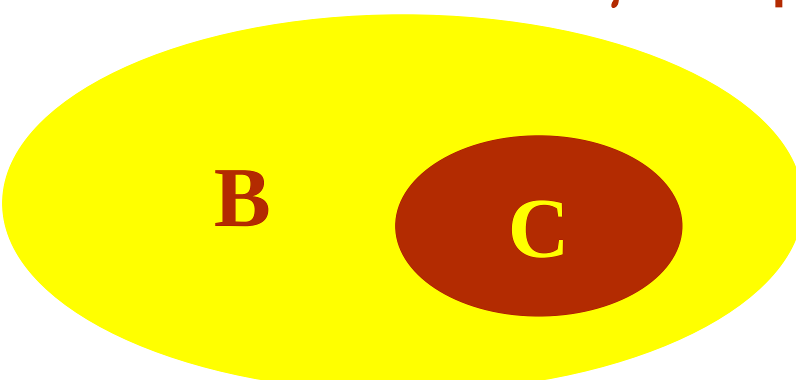
new users



surviving users

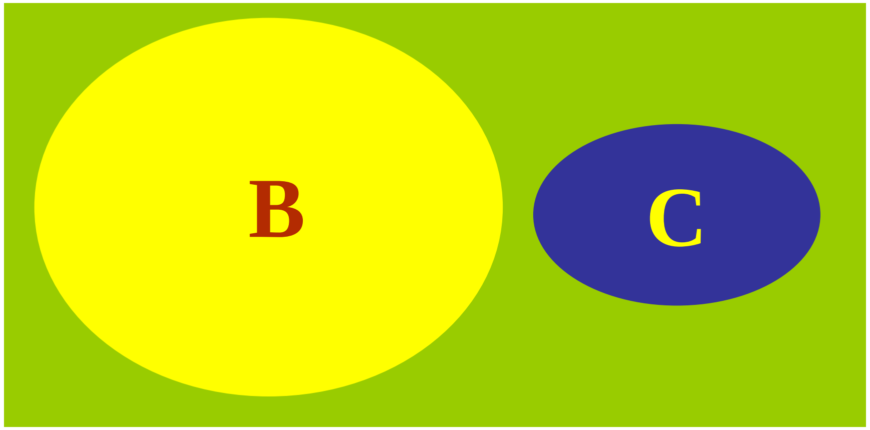
$$f_{\mathbf{S}_t|\mathbf{X}_{t-1}}(\mathbf{C} \mid \mathbf{B}) = \begin{cases} \mu^{|\mathbf{C}|}(1-\mu)^{|\mathbf{B}|-|\mathbf{C}|}, & \mathbf{C} \subseteq \mathbf{B} \\ 0, & \mathbf{C} \nsubseteq \mathbf{B} \end{cases}$$

μ = probability of persistence



new users

$$f_{\mathbf{N}_t|\mathbf{X}_{t-1}}(\mathbf{C} \mid \mathbf{B}) = \begin{cases} \alpha^{|\mathbf{C}|}(1-\alpha)^{K-|\mathbf{B}|-|\mathbf{C}|}, & \mathbf{C} \cap \mathbf{B} = \emptyset \\ 0, & \mathbf{C} \cap \mathbf{B} \neq \emptyset \end{cases}$$



α = activity factor

surviving users + new users

Derive the belief density of $\mathbf{X}_t = \mathbf{S}_t \cup \mathbf{N}_t$ through the "generalized convolution"

$$
\begin{aligned}
f_{\mathbf{X}_t|\mathbf{X}_{t-1}}(\mathbf{C} \mid \mathbf{B}) \\
&= \sum_{\mathbf{W} \subseteq \mathbf{C}} f_{\mathbf{S}_t|\mathbf{X}_{t-1}}(\mathbf{W} \mid \mathbf{B})\, f_{\mathbf{N}_t|\mathbf{X}_{t-1}}(\mathbf{C} \setminus \mathbf{W} \mid \mathbf{B}) \\
&= f_{\mathbf{S}_t|\mathbf{X}_{t-1}}(\mathbf{C} \cap \mathbf{B}) f_{\mathbf{N}_t|\mathbf{X}_{t-1}}(\mathbf{C} \setminus (\mathbf{C} \cap \mathbf{B}))
\end{aligned}
$$

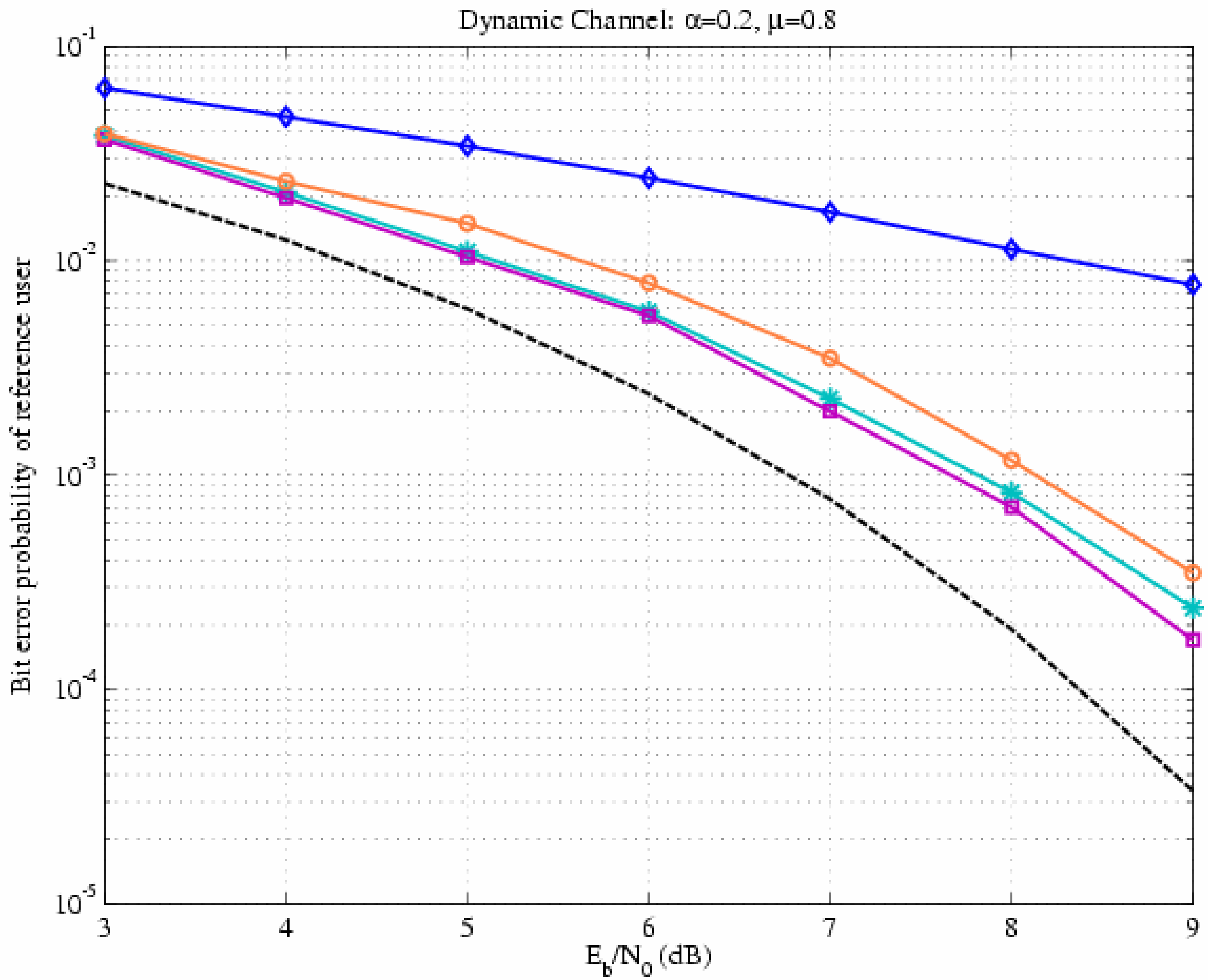


Figure 3: *Bit error probability of the reference user in a multiuser system with 2 interferers, following a dynamic model described above with $\alpha = 2$ and $\mu = 0.8$. Line with diamond markers: Classic multiuser ML detection, assuming that all users are active. Line with circle markers: MAP detection based on the knowledge of α alone. Line with star markers: causal RST detector, based on Bayes recursions. Line with square markers: Viterbi RST detector. Dashed curve: Single-user bound.*

lesson learned

- MUD receivers must know the number of interferers, otherwise performance is impaired.
- Introducing a priori information about the number of active users improves MUD performance and robustness.
- A priori information may include *activity factor.*
- A priori information may also include a model of users' motion.

detection and estimation

- In addition to detecting the number of active users and their data, one may want to estimate their parameters (e.g., their power)

- A Markov model of power evolution is needed

effect of fading

$$P_t = R_t^2 P$$

$$P_{t+1} = \frac{R_{t+1}^2}{R_t^2} P_t = \gamma_t P_t$$

$$\gamma_t \triangleq R_{t+1}^2 / R_t^2$$

effect of motion

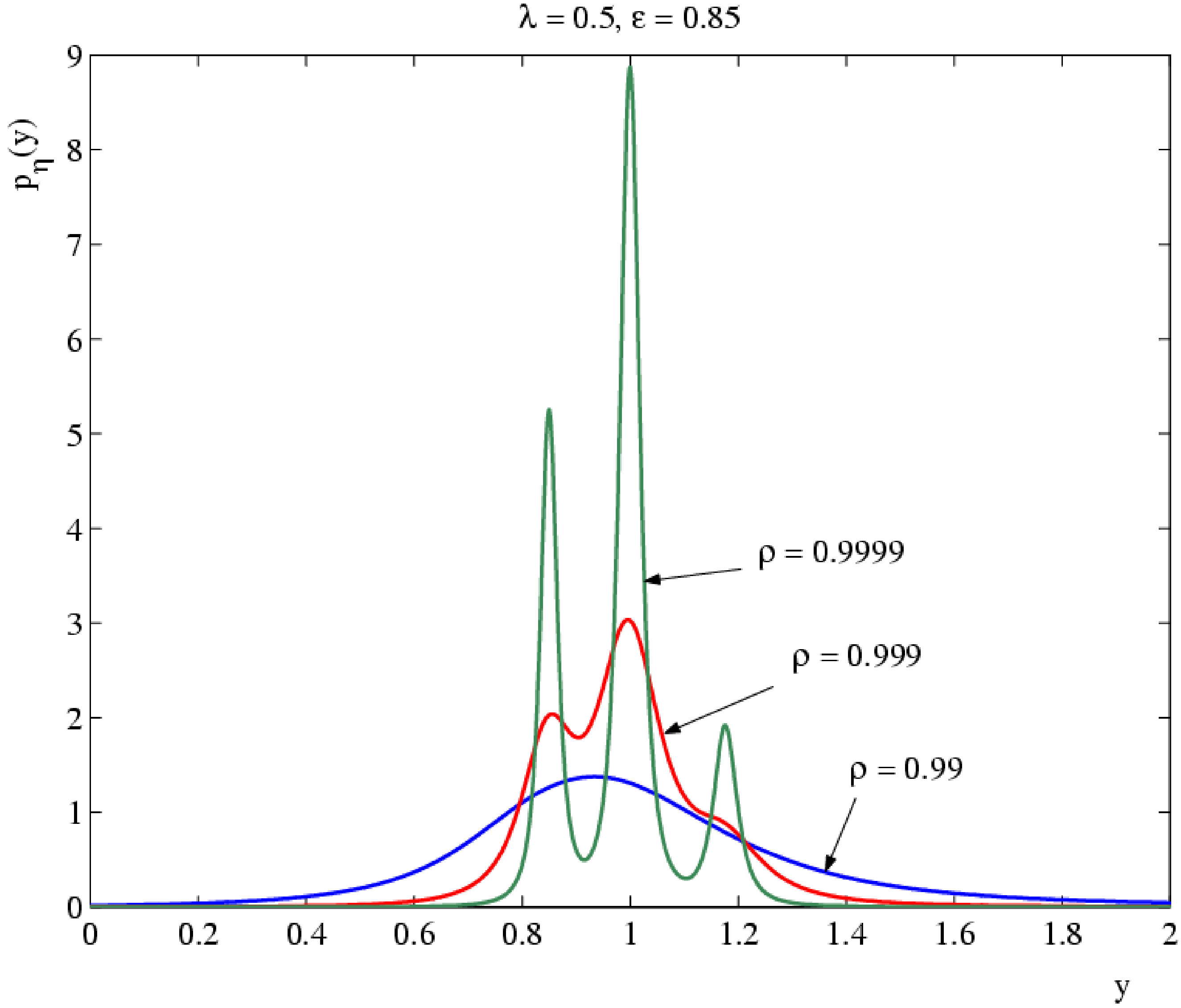
$$P_{t+1} = \delta_t P_t$$

$$\delta_t = \begin{cases} 1 & \text{with probability } 1-\lambda \\ \varepsilon & \text{with probability } \lambda/2 \\ \varepsilon^{-1} & \text{with probability } \lambda/2 \end{cases}$$

joint effects

$$P_{t+1} = \gamma_t \delta_t P_t = \eta_t P_t$$

$$\eta_t \triangleq \gamma_t \delta_t$$



pdf of η for Rayleigh fading

APPLICATIONS TO WIRELESS COMMUNICATIONS: MULTIPATH CHANNEL ESTIMATION

channel model (OFDM)

channel frequency response

$$\mathbf{y}_p = \mathbf{D}_p \overbrace{\mathbf{F}_p \mathbf{h}_p} + \mathbf{z}_p$$

observed signal

discrete time

training sequence
(diagonal matrix)

delays

path gains

additive noise

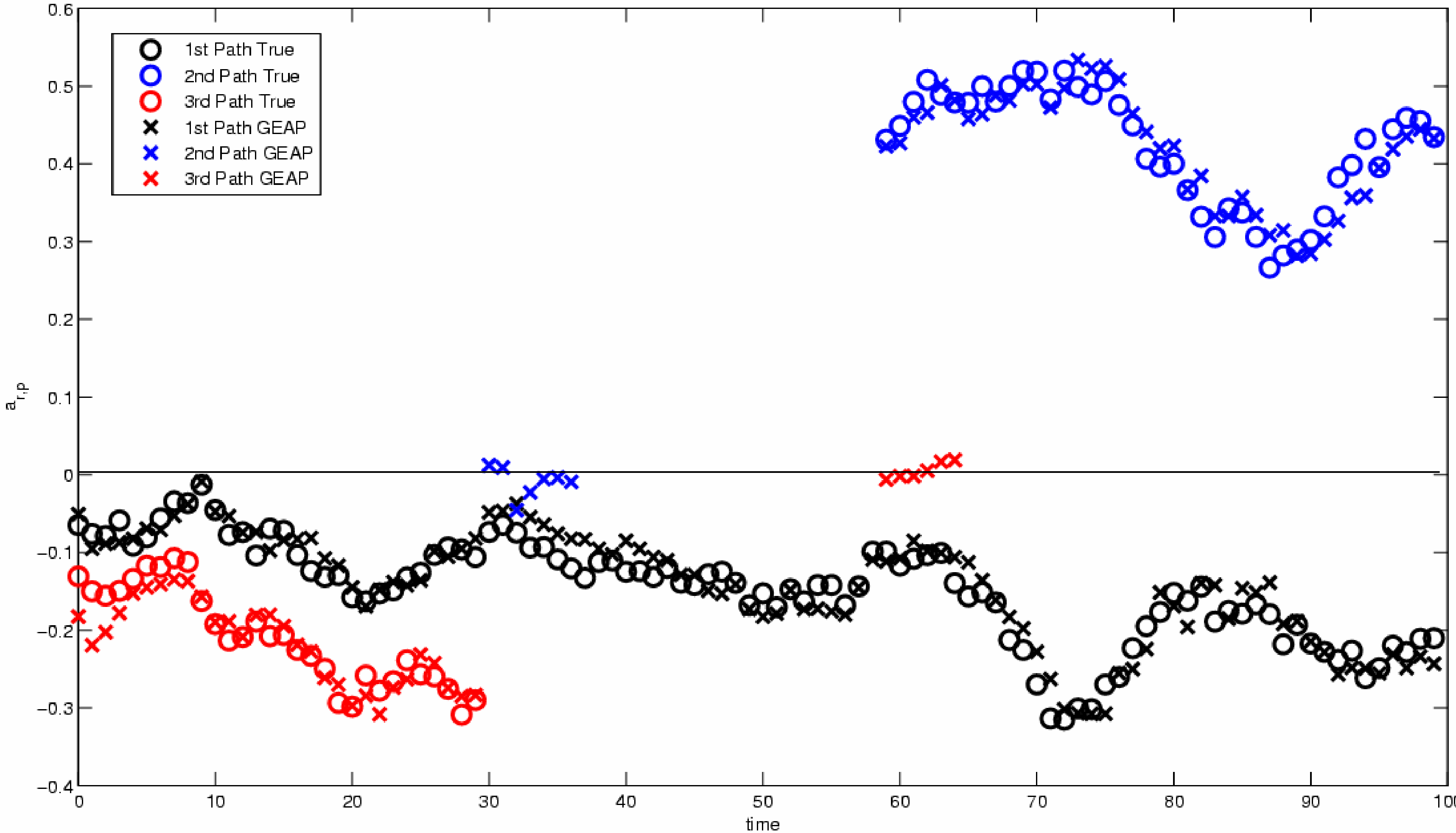
random sets

The channel response at time t is modeled by the random set

$$\mathbf{X}_t = \{\mathbf{h}_1^{(t)}, \ldots, \mathbf{h}_k^{(t)}\}, \quad k = 1, \ldots, L_{\max}$$

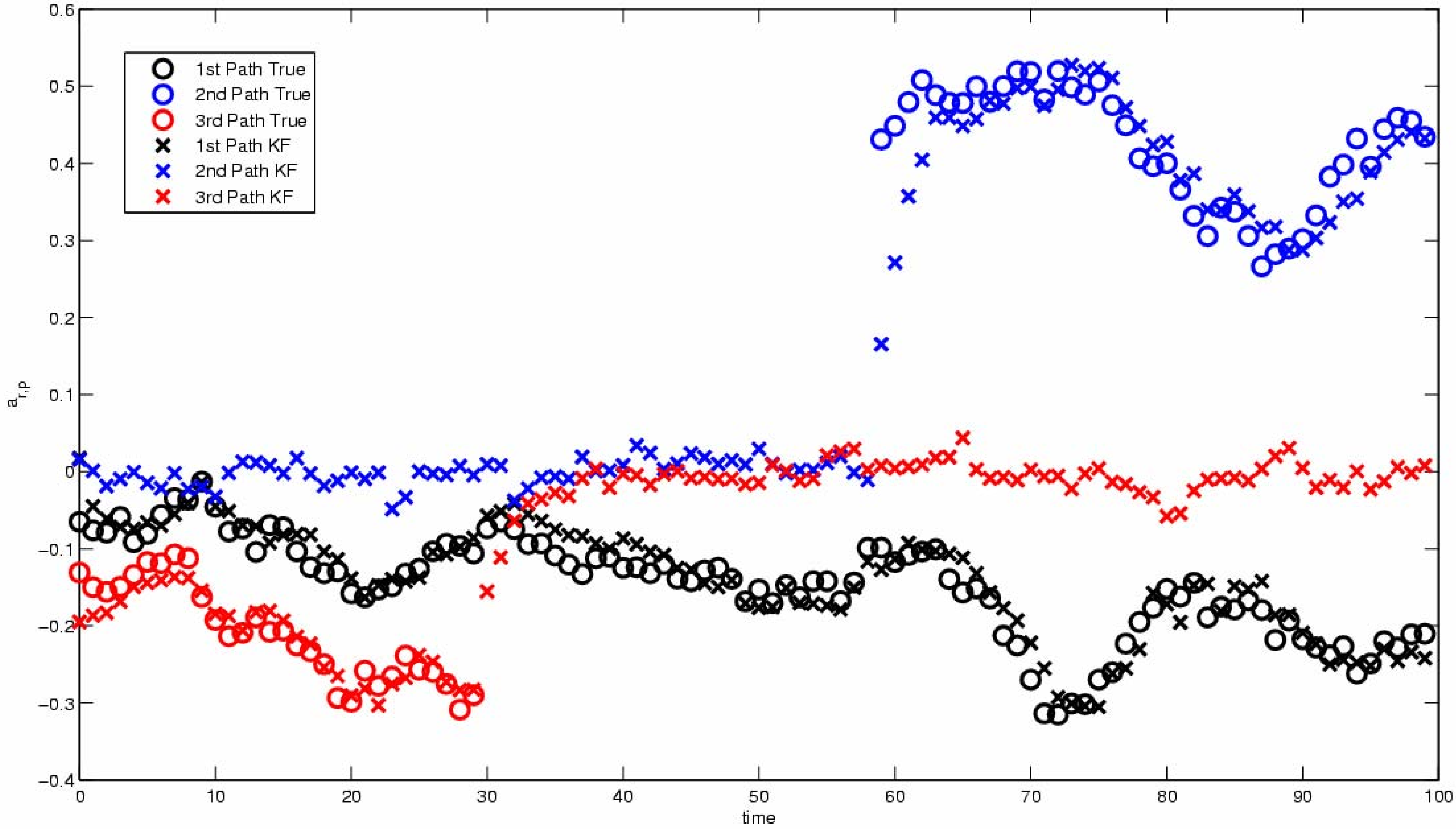
where L_{max} is the maximum number of active interferers

performance of GMAP-III



Real part, SNR = 20 dB

performance of Kalman filter



Real part, SNR = 20 dB

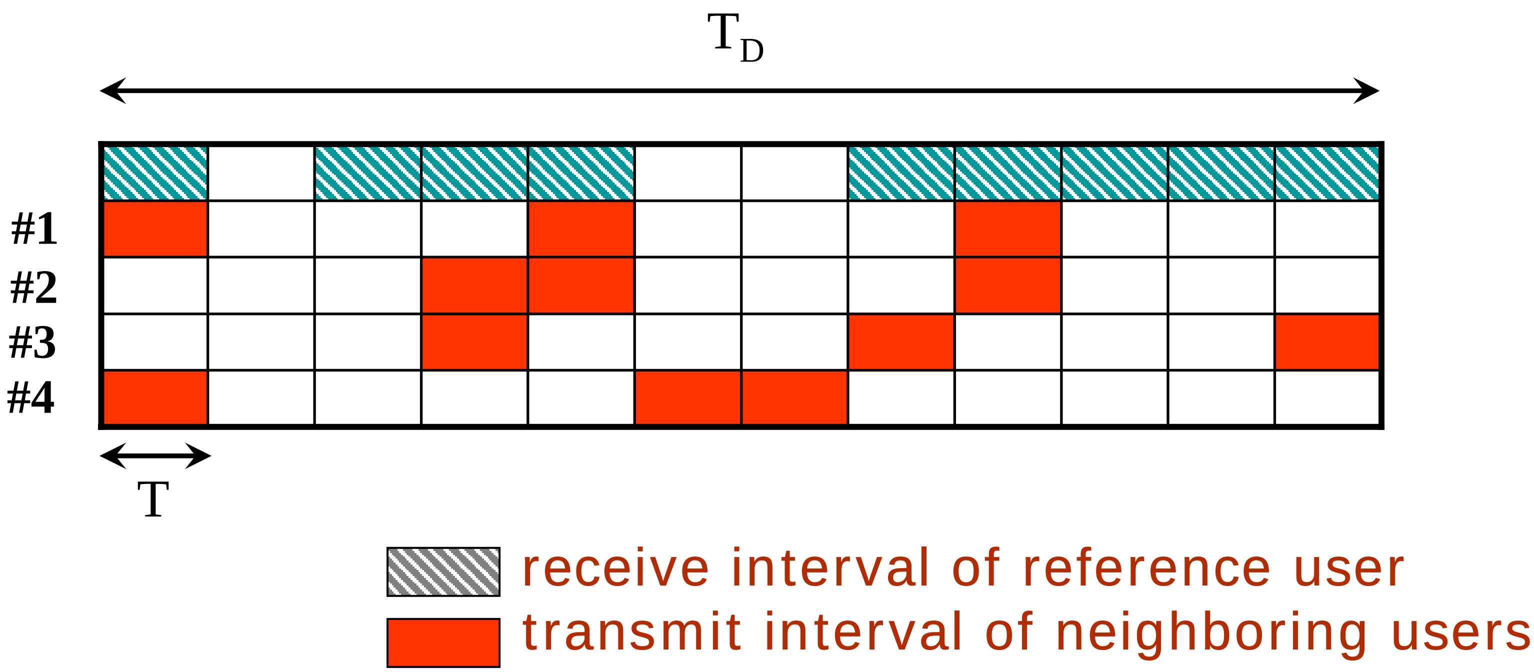
APPLICATIONS TO WIRELESS COMMUNICATIONS: NEIGHBOR DISCOVERY

neighbor discovery

- In a wireless network, neighbor discovery (ND) is the detection of all neighbors with which a given reference node may communicate directly.

- ND may be the first algorithm run in a network, and the basis of medium access, clustering, and routing algorithms.

neighbor discovery



- Structure of a discovery session

neighbor discovery

Signal collected from all potential neighbors during receiving slot t :

$$\mathbf{y}_t = \sum_k \alpha_k \mathbf{s}_k + \mathbf{n}_t$$

extended to a random number of users

amplitude of user k

signature of user k

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